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NONLINEAR LAWS OF DRY FRICTION IN CONTACT PROBLEMS OF LINEAR THEORY OF ELASTICITY

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A problem of elastic body deformation with conditions of dry friction imposed at the boundary is considered. Various friction laws are studied, including linear one-parameter and nonlinear two-parameter laws. A general view of a nonlinear function with two constants is suggested, which determines the friction force as a function of normal pressure. The problem of elastic plate compression by rough infinite plates for a variety of friction conditions on the contact surfaces is solved. The plate equations are employed, which make it possible to specify arbitrary conditions on the front faces without reducing the order of differential equations. The unknown boundary of ideal contact zones and sliding zones is determined. Solutions obtained by using various friction conditions on the contact surfaces are compared.

1. Formulation of the Problem. Let us consider an elastic body of volume V bounded by the surface S, with friction conditions being specified on its portion S_τ. Let $\boldsymbol{P}$ be the vector of external forces acting on the surface S. Let us resolve $\boldsymbol{P}$ into components tangential $\boldsymbol{\tau}$ and normal σ_n to the surface S:

$$\boldsymbol{P} = \boldsymbol{\tau} + \sigma_n \boldsymbol{n}, \tag{1}$$

where $\boldsymbol{n}$ is the unit vector of the outer normal.

The law of dry friction (friction without lubrication) implies that such conditions are specified on the contact surface S_τ, for which the value of tangential stresses depends only on the value of normal stresses, i.e., tangential and normal components of the vector of external forces (1) are related as

$$|\boldsymbol{\tau}| = \tau(|\sigma_n|, \gamma_i), \qquad i = \overline{1, N}. \tag{2}$$

Here, τ is some positive function, γ_i is a set of parameters characterizing the roughness of the contact surface, strength properties of the contact bodies, etc. The function $\tau(|\sigma_n|, \gamma_i)$ in (2) specifies the particular dry friction law on the contact surface.

Among the simplest examples of mathematical simulation of the friction process, there are one-parameter ($N = 1$) laws

$$\tau(|\sigma_n|, \tau_s) = \tau_s; \tag{3}$$

$$\tau(|\sigma_n|, k) = k|\sigma_n|. \tag{4}$$

The first relation is the law of a constant friction force, and the second is the Amonton–Coulomb law of friction. The parameter τ_s can be interpreted as the yield stress of the contacting layer. The friction laws (3) and (4) have been theoretically investigated [1] and are widely used in calculations. However, when relations (3) and (4) are used, the results are often unsatisfactory. Thus, the law of a constant friction force (3) provides a sufficiently accurate description of external friction in the zones of high normal stresses but can introduce a large error for the areas with normal stresses approaching zero. The Amonton–Coulomb friction law (4) is applicable for low normal pressures. As is noted in [2], the relationship between the specific friction force and normal specific pressure is more complex.

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It would be feasible to use two-parameter friction functions ($N = 2$) depending simultaneously on the parameters τ_s and k, as such a dependence is free from the above-mentioned disadvantages:

$$|\boldsymbol{\tau}| = \tau(|\sigma_n|, k, \tau_s) = \tau_s g(\eta), \qquad \eta = k|\sigma_n|/\tau_s. \tag{5}$$

As an example of a two-parameter dependence, we can name the Prandtl–Il'yushin friction law [3, 4], to which there corresponds a piecewise linear function

$$g_c(\eta) = 0.5(\eta + 1 - |\eta - 1|), \qquad \eta \in [0, \infty). \tag{6}$$

In zones with low normal stresses ($\eta < 1$), from (5) there follows the Amonton–Coulomb friction law (3), and in high-pressure areas ($\eta \geqslant 1$), there follows the law of a constant friction force (4).

The distribution of contact stresses can be described more completely by relations based on nonlinear functions $g(\eta)$, $\eta \in [0, \infty)$ satisfying the conditions

$$\begin{gathered} g(0) = 0, \qquad g(\infty) = 1, \\ \frac{dg}{d\eta}(0) = 1, \qquad \frac{dg}{d\eta}(\eta) \geqslant 0, \qquad \eta \in [0, \infty). \end{gathered} \tag{7}$$

The function $g(\eta)$ is positive, monotonously increasing, with an asymptote at $\eta \to \infty$. For every $\eta \in [0, \infty)$ there holds the inequality

$$g(\eta) \leqslant g_c(\eta) \leqslant 1. \tag{8}$$

Conditions (7) define a convex set of functions G, where each function specifies the nonlinear two-parameter friction law (5). Elements of this set are, for example, the functions

$$g(\eta) = \eta/(1 + \eta), \qquad g(\eta) = 1 - \mathrm{e}^{-\eta}, \qquad g(\eta) = \eta/\sqrt{1 + \eta^2}, \qquad g(\eta) = \tanh \eta.$$

The first two dependences coincide with the Tirion and Bartenev–Lavrent'ev dependences obtained in processing the experimental data of [5] to accuracy of notation.

Similarly to (1), let us resolve the displacement vector at the boundary S into tangential ($\boldsymbol{u}_\tau$) and normal (u_n) components:

$$\boldsymbol{u} = \boldsymbol{u}_\tau + u_n \boldsymbol{n}.$$

Let us formulate the two-parameter law of dry friction (6) in the form similar to that of [1]:

— on the surface, $S_\tau \quad \sigma_n \leqslant 0$, $u_n = u_*$;

— within the ideal contact zone,

$$|\boldsymbol{\tau}| < \tau_s g(\eta) \qquad \Rightarrow \qquad \boldsymbol{u}_\tau = 0; \tag{9}$$

— in the sliding zone,

$$|\boldsymbol{\tau}| = \tau_s g(\eta) \qquad \Rightarrow \quad \exists \lambda \geqslant 0 \quad \boldsymbol{u}_\tau = -\lambda \boldsymbol{\tau}. \tag{10}$$

It follows from (10) that the function λ in the sliding zone has the form

$$\lambda = |\boldsymbol{u}_\tau|/(\tau_s g(\eta)) \tag{11}$$

and

$$\boldsymbol{\tau} = -\mathrm{sign}\,(\boldsymbol{u}_\tau) \tau_s g(\eta), \qquad \eta = k|\sigma_n|/\tau_s. \tag{12}$$

Relations (9)–(12) for a specified function $g(\eta)$ satisfying conditions (7) determine a two-parameter nonlinear friction law with an unknown boundary of ideal contact zones and sliding zones.

In order to compare one- and two-parameter friction laws, let us consider the problem of elastic plate compression by rough infinite plates. Three variants of friction on the contact surfaces are assumed: the law of a constant tangential stress, Coulomb's friction law, and the Prandtl–Il'yushin law with a piecewise-linear friction function. These conditions were chosen for the following reasons. The law of a constant friction force and Coulomb's friction law are classical and most widely used in calculations, and the piecewise linear dependence is a combination of the first two laws. At the same time, the piecewise-linear friction function depends on two parameters and possesses a limit property [see inequality (8)] for the set G of friction functions satisfying conditions (7).

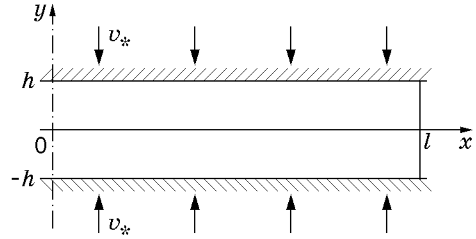

Fig. 1

2. Plate Compression by Rough Infinite Plates. A plate of thickness $2h$ and width $2l$ under conditions of plane strain is compressed by rigid rough plates with a specified displacement v_* (Fig. 1).

Let us use equations of elastic strain of plates and shells [6, 7], based on the series expansion of the sought functions in Legendre polynomials. These equations admit specification of arbitrary conditions on the front faces without reducing the order of differential equations. One can specify stresses, displacements, or combined conditions. This allows a correct formulation of conjugation conditions at the interface of zones of sliding and adhesion (ideal contact). Such equations were used in solving the problems of elastic strain of layered bodies [8–10].

The stresses are approximated by truncated series in Legendre polynomials $P_k(\xi)$ $(\xi = y/h)$:

$$2h\sigma_x = N + 3MP_1(\xi)/h, \quad \sigma_y = p_0 + \Delta p P_1(\xi), \quad 2h\sigma_{xy} = Q + 2h\Delta q P_1(\xi) + (2hq_0 - Q)P_2(\xi),$$
$$\Delta p = 0.5(p^+ - p^-), \quad p_0 = 0.5(p^+ + p^-), \quad \Delta q = 0.5(q^+ - q^-), \quad q_0 = 0.5(q^+ + q^-). \tag{13}$$

Here $N = \int\limits_{-h}^{h} \sigma_x\,dy$ is the force, $M = \int\limits_{-h}^{h} \sigma_x y\,dy$ is the moment, $Q = \int\limits_{-h}^{h} \sigma_{xy}\,dy$ is the transverse shear force, and $p^\pm$ and $q^\pm$ are the normal and tangential stresses on the contact surfaces $(\xi = \pm 1)$.

The displacements and strains are approximated by truncated series in Legendre polynomials

$$u_x = u + \psi P_1(\xi) + (u_0 - u)P_2(\xi) + (\Delta u - \psi)P_3(\xi), \quad u_y = v + \Delta v P_1(\xi) + (v_0 - v)P_2(\xi),$$

$$e_x = \frac{du}{dx} + \frac{d\psi}{dx} P_1(\xi), \qquad e_y = \frac{1}{h}\Delta v + \frac{3}{h}(v_0 - v)P_1(\xi),$$

$$e_{xy} = \frac{dv}{dx} + \frac{1}{h}\Delta u + \frac{3}{h}(u_0 - u)P_1(\xi) + \frac{5}{h}(\Delta u - \psi)P_2(\xi), \tag{14}$$

$$\Delta u = 0.5(u^+ - u^-), \quad u_0 = 0.5(u^+ + u^-), \quad \Delta v = 0.5(v^+ - v^-), \quad v_0 = 0.5(v^+ + v^-).$$

Here $u = \frac{1}{2}\int\limits_{-1}^{1} u_x\,d\xi$ and $v = \frac{1}{2}\int\limits_{-1}^{1} u_y\,d\xi$ are displacements averaged through the thickness, $\psi = \frac{3}{2}\int\limits_{-1}^{1} u_x \xi\,d\xi$ is the rotation angle of the normal vector to the midplane $y = 0$, and $v^\pm$ and $u^\pm$ are the normal and tangential displacements on the contact surfaces $(\xi = \pm 1)$.

Unknown functions entering into the polynomial coefficients in formulas (13) and (14) are found from a system comprising the following equations:

— equations of equilibrium

$$\frac{dN}{dx} + 2\Delta q = 0, \qquad \frac{dM}{dx} - Q + 2hq_0 = 0, \qquad \frac{dQ}{dx} + 2\Delta p = 0; \tag{15}$$

— differential equations obtained from Hooke's law

$$\frac{du}{dx} = \frac{N}{2hE_*} - \nu_*\frac{p_0}{E_*}, \quad \frac{d\psi}{dx} = \frac{3M}{2h^2E_*} - \nu_*\frac{\Delta p}{E_*}, \quad \frac{dv}{dx} + \frac{\Delta u}{h} = \frac{Q}{2h\mu}; \tag{16}$$

— algebraic equations obtained from Hooke's law

$$u_0 - u = \frac{h}{3\mu}\,\Delta q, \quad \Delta u - \psi = \frac{h}{5\mu}\Big(q_0 - \frac{Q}{2h}\Big), \quad \Delta v = h\,\frac{p_0}{E_*} - \nu_*\,\frac{N}{2E_*},$$

$$v_0 - v = h\,\frac{\Delta p}{3E_*} - \nu_*\,\frac{M}{2hE_*}, \quad E_* = \frac{E}{1-\nu^2}, \quad \nu_* = \frac{\nu}{1-\nu}. \tag{17}$$

Here E is Young's modulus, μ is the shear modulus, and ν is Poisson's ratio.

The system of ordinary differential equations (15) and (16) for the unknown functions N, M, Q, u, ψ, and v is of the sixth order.

The unknown functions Δu, u_0, Δv, v_0, Δp, p_0, Δq, and q_0 are found from Eqs. (17) and the boundary conditions on the contact surfaces $\xi = \pm 1$.

Since the problem is symmetric about the plane $y = 0$, the following relations for stresses hold on the contact surfaces $y = \pm h$ ($\xi = \pm 1$): $q^+ = -q^- = q$ and $p^+ = p^- = p$. Accordingly, for the displacements, we have $v^+ = -v^- = -v_*$ and $u^+ = u^- = w$. Substitution of these relation into formulas (13) and (14) yields

$$q_0 = 0, \qquad \Delta q = q, \qquad p_0 = p, \qquad \Delta p = 0,$$

$$v_0 = 0, \qquad \Delta v = -v_*, \qquad u_0 = w, \qquad \Delta u = 0. \tag{18}$$

Here q, p, and w are the unknown functions and v_* is the specified plate displacement.

For $x = 0$ (symmetry plane) and $x = l$ (free surface), we have the following boundary conditions:

$$u = 0, \qquad \psi = 0, \qquad Q = 0 \qquad \text{for} \quad x = 0,$$

$$N = 0, \qquad M = 0, \qquad Q = 0 \qquad \text{for} \quad x = l. \tag{19}$$

Substitution of (18) into system (15)–(17) allowing for the boundary conditions (19) yields $M = 0$, $Q = 0$, $\psi = 0$, and $v = 0$. There remain unknown functions N, u, p, q, and w calculated from the equations

$$\frac{dN}{dx} + 2q = 0, \qquad \frac{du}{dx} - \frac{N}{2hE_*} + \frac{\nu_* p}{E_*} = 0, \tag{20}$$

$$u - w + hq/(3\mu) = 0, \qquad v_* + hp/E_* - \nu_* N/(2E_*) = 0$$

with the boundary conditions

$$u = 0 \quad \text{for} \quad x = 0, \qquad N = 0 \quad \text{for} \quad x = l. \tag{21}$$

The fifth equation required to close system (20) follows from the chosen friction law.

Let us consider three variants of boundary conditions on the contact surfaces, i.e., ideal contact ($w = 0$), constant tangential stress ($q = -\tau_s$), and Coulomb's friction law ($q = kp$). In the first case, equations (20) must be supplemented by the condition

$$w = 0. \tag{22}$$

The resulting system (20), (22) has the solution

$$u = A_1 \sinh\,(\alpha x), \quad \alpha = \frac{1}{h}\sqrt{\frac{3(1-\nu_*)}{2}}, \quad N = \frac{2hE_*}{1-\nu_*^2}\Big(\alpha A_1 \cosh\,(\alpha x) - \frac{\nu_* v_*}{h}\Big),$$

$$q = -\frac{3E_*}{2h(1+\nu_*)}\,A_1 \sinh\,(\alpha x), \qquad p = \frac{E_*}{1-\nu_*^2}\Big(\alpha \nu_* A_1 \cosh\,(\alpha x) - \frac{v_*}{h}\Big). \tag{23}$$

This solution was obtained with allowance for the assumption that the ideal contact zone adjoins the axis ($x = 0$).

For the case of a constant tangential stress, the following equality must be supplemented:

$$q = -\tau_s. \tag{24}$$

System (20), (24) has a solution

$$N = 2\tau_s x + A_2, \qquad u = (1-\nu_*^2)(\tau_s x^2 + A_2 x)/(2hE_*) + \nu_* v_* x/h + B_2,$$

$$q = -\tau_s, \qquad p = \nu_*(2\tau_s x + A_2)/(2h) - E_* v_*/h. \tag{25}$$

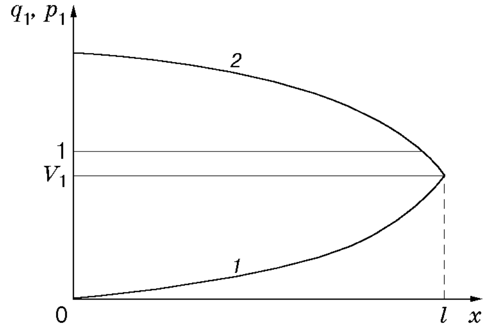

Fig. 2

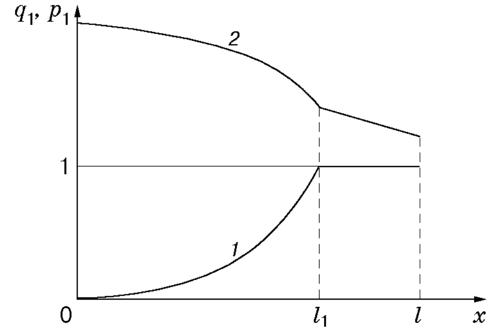

Fig. 3

For the case of Coulomb's friction law the following condition is added:

$$q = kp. \tag{26}$$

System (20), (26) has the solution

$$N = A_3 \exp\Big(-\frac{k\nu_* x}{h}\Big) + 2\,\frac{E_*}{\nu_*}\,v_*, \quad u = -\frac{1-\nu_*^2}{2k\nu_* E_*}\,A_3 \exp\Big(-\frac{k\nu_* x}{h}\Big) + \frac{v_*}{\nu_* h}\,x + B_3,$$
$$q = \frac{k\nu_*}{2h}\,A_3 \exp\Big(-\frac{k\nu_* x}{h}\Big), \qquad p = \frac{\nu_*}{2h}\,A_3 \exp\Big(-\frac{k\nu_* x}{h}\Big). \tag{27}$$

The solution of every contact problem considered below is a combination of the solutions obtained. The unknown constants are found from the boundary conditions (21) and conjugation conditions at the interface of the domains with different friction conditions:

$$[u] = 0, \qquad [N] = 0, \qquad [q] = 0 \qquad \text{for} \quad x = l_i \tag{28}$$

(square brackets denote the function jump).

2.1. *Law of a Constant Friction Force.* For the displacement of the plates, at first there emerges one zone of ideal contact [solution of (22)]. The constant A_1 is obtained from the boundary condition on the free surface (21)

$$A_1 = \nu_* v_*/(\alpha h \cosh\,(\alpha l)).$$

As compression intensifies, contact stresses increase. For $v_* = v_{p1}$, where $v_{p1} = (\tau_s(1-\nu_*^2)/(\nu_* E_* \alpha)) \times \coth\,(\alpha l)$, the contact tangential stress $|q|$ on the edge of the plate for $x = l$ reaches the value τ_s, and a second zone emerges, i.e., the sliding zone.

Let us introduce the following dimensionless quantities:

$$q_1 = |q|/\tau_s, \qquad p_1 = |p|h/(E_* v_{p1}), \qquad V_1 = v_*/v_{p1}.$$

Figure 2 shows the distribution of dimensionless contact stresses q_1 (curve 1) and p_1 (curve 2).

For $v_* > v_{p1}$, the problem solution is a combination of two solutions (23) and (25). The zone of ideal contact adjoins the y axis, and the sliding zone is adjoint to the free surface $x = l$. Let $x = l_1$ be the equation of an unknown interface between the sliding and adhesion zones. Constants entering in the solutions (23) and (25) are calculated from the boundary conditions (21) and conjugation conditions (28).

The constants A_1, A_2, and B_2 take the following values:

$$A_1 = \frac{(1-\nu_*^2)\tau_s(l_1 - l) + \nu_* v_* E_*}{\alpha h E_* \cosh\,(\alpha l_1)}, \quad A_2 = -2\tau_s l, \quad B_2 = \sinh\,(\alpha l_1) A_1 - \frac{\tau_s l_1 (l_1 - 2l)}{2hE_*} - \frac{\nu_* v_* l_1}{h}.$$

In order to determine the interface $x = l_1$ ($0 < l_1 < l$) between the ideal contact and sliding zones, we obtain a transcendental equation

$$\frac{(1-\nu_*^2)\tau_s}{\nu_* E_*}\Big(\frac{\coth\,(\alpha l_1)}{\alpha} + l - l_1\Big) = v_*.$$

With an increase in v_*, the sliding zone also increases and the interface $x = l_1$ is shifted to the center.

Figure 3 shows the distribution of dimensionless contact stresses q_1 (curve 1) and p_1 (curve 2).

An analysis of the solution obtained enables us to draw the following conclusions. Under plate displacement, with an increase in load, the friction force attains the limiting value and then remains constant, which adequately reflects the characteristic features of the external friction process [5]. At the same time, for low values of the normal contact stress, it follows from the problem solution that, for every arbitrarily small value of τ_s (contact surface is almost ideally smooth), there is a range of initial displacements $0 < v_* < v_{p1}$, for which a ideal contact (adhesion) takes place over the entire plate surface, which disagrees with experimental results.

2.2. *Amonton–Coulomb Friction Law.* When the Amonton–Coulomb friction law is used, there emerge two zones: the ideal contact zone [solution (23)] adjoining the y axis and the sliding zone [solution (27)] adjoint to the free surface $x = l$. The unknown constants entering in these solutions are found from the boundary conditions (21) and conjugation conditions (28) at the interface of the zones $x = l_2$.

The constants A_1, A_3, and B_3 are calculated by the formulas

$$A_1 = \frac{2k(1+\nu_*)}{3\sinh(\alpha l_2)} \exp\Big(\frac{k\nu_*(l-l_2)}{h}\Big), \qquad A_3 = -\frac{2E_* v_*}{\nu_*} \exp\Big(\frac{k\nu_* l}{h}\Big),$$

$$B_3 = v_*\Big[\Big(\frac{2k(1+\nu_*)}{3} - \frac{1-\nu_*^2}{k\nu_*^2}\Big) \exp\Big(\frac{k\nu_*(l-l_2)}{h}\Big) - \frac{l_2}{\nu_* h}\Big].$$

The interface between the adhesion and sliding zones $x = l_2$ is obtained from the transcendental equation

$$(1-\nu_*^2)\exp\Big(\frac{k\nu_*(l-l_2)}{h}\Big)\Big(\frac{k\nu_*\coth(\alpha l_2)}{h\alpha} + 1\Big) = 1. \tag{29}$$

It follows from (29) that l_2 does not depend on v_*, i.e., and the position of the interface between the adhesion and sliding zones does not change under plate displacement.

The maximum value of the contact tangential stress is reached at the interface between the zones $x = l_2$:

$$\max|q| = (v_* k E_*/h)\exp(k\nu_*(l-l_2)/h). \tag{30}$$

Under plate displacement, the value of $\max|q|$ increases and reaches τ_s for $v_* = v_{p2}$, where $v_{p2} = h\tau_s \exp(k\nu \times (l_2 - l)/h)/(kE)$.

It follows from (30) that, even for small values of v_* (small plates displacement), the value of $\max|q|$ may exceed the yield stress of the contact layer τ_s.

Let us introduce the following dimensionless quantities:

$$q_2 = q_1 = |q|/\tau_s, \qquad p_2 = |p|h/(E_* v_{p2}), \qquad V_2 = v_*/v_{p2}.$$

Figure 4 shows the distribution of dimensionless contact tangential stress q_2 (curve 1) and normal stress p_2 (curve 2) for $V_2 < 1$.

When the friction coefficient k changes from 0 to k_p where $k_p = \nu_* h\alpha \tanh(\alpha l)/(1-\nu_*^2)$, the values of l_2 change from 0 to l, i.e., as k increases, the interface is shifted from the center to the edge of the plate.

For $k \geqslant k_p$, we have $l_2 > l$ and the interface between the zones goes beyond the plate. Consequently, for every v_*, there exists one zone, i.e., the zone of ideal contact. In this case, the problem solution coincides with the solution from Sec. 2.1.

The following conclusions can be drawn from the analysis of the results obtained. The Amonton–Coulomb friction law adequately describes the process of external friction at the initial stage of plate displacement when the contact normal stresses are rather low. However, the application of this law to the case of the further plate compression may yield physically erroneous results: the tangential contact stress can exceed the yield stress of the contact layer. In addition, under displacement of the plates, the position of the interface between the zone of ideal contact and the sliding zone does not change, and no sliding zone is formed for large values of the friction coefficient.

2.3. *The Prandtl–Il'yushin Law with a Piecewise-Linear Friction Function.* Let us consider two variants of solutions: 1) for $k < k_p$; 2) $k \geqslant k_p$.

In the first case, the contact interaction at the initial stage obeys the Amonton– Coulomb friction law. The solution coincides with that obtained in Sec. 2.2. Initially, two zones emerge: the ideal contact and sliding zones.

Under plate displacement, the corresponding value of $\max|q|$ [see (30)] increases, and reaches the value of τ_s for $v_* = v_{p2}$. At this moment, the third zone emerges, i.e., the zone of a constant tangential stress.

Therefore, for $v_* > v_{p2}$, the problem solution is a combination of three solutions: (23), (25), and (27). The ideal contact zone adjoins the y axis, it is followed by the zone of a constant friction force, and then comes the sliding zone adjoint to the free surface $x = l$, which obeys the Amonton–Coulomb friction law.

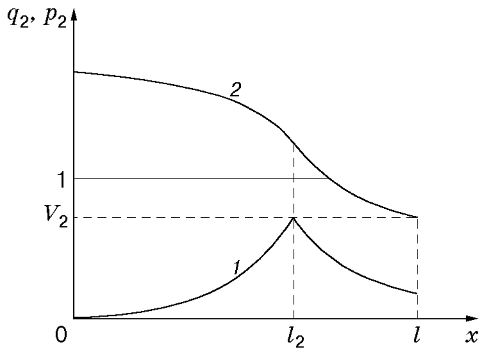

Fig. 4

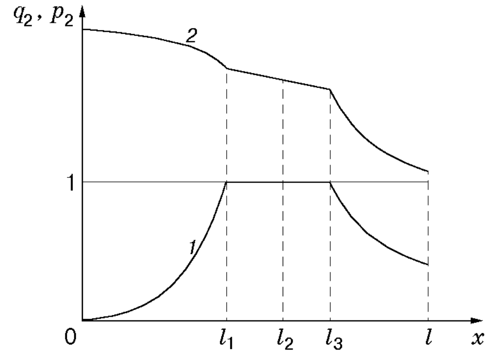

Fig. 5

There are two unknown interfaces: the first one at $x = l_1$ separates the ideal contact zone from the zone with contact interaction obeying the law of a constant friction force, and the second one at $x = l_3$ separates the zone of a constant friction force from the zone obeying the Amonton–Coulomb friction law. The constants entering in the solutions (23), (25), and (27) are calculated from the boundary conditions (21) and conjugation conditions (28) at the points $x = l_1$ and $x = l_3$.

The constants A_1, A_2, B_2, A_3, and B_3 are calculated by the formulas

$$A_1 = \frac{1}{\alpha h \cosh\,(\alpha l_1)} \Big\{ \frac{v_*}{\nu_*} + (1-\nu_*^2) \Big[\frac{\tau_s}{E_*}(l_1 - l_3) - \frac{v_*}{\nu_*} \exp\Big(\frac{kv_*}{h}(l - l_3)\Big)\Big]\Big\},$$

$$A_2 = -2\tau_s l_3 + 2\,\frac{E_* v_*}{\nu_*} \exp\Big(\frac{kv_*}{h}(l - l_3)\Big), \qquad A_3 = -2\,\frac{E_* v_*}{\nu_*} \exp\Big(\frac{kv_*}{h}\, l\Big),$$

$$B_2 = \sinh\,(\alpha l_1) A_1 - \frac{1-\nu_*^2}{2hE_*}(\tau_s l_1^2 + A_2 l_1) - \frac{v_* \nu_*}{h}(l_3 - l_1),$$

$$B_3 = \sinh\,(\alpha l_1) A_1 + \frac{1-\nu_*^2}{2hE_*}(\tau_s (l_3^2 - l_1^2) + A_2 (l_3 - l_1)) + \frac{v_* \nu_*}{h}(l_3 - l_1) + \frac{(1-\nu_*^2) k E_*}{\nu_*^2 \tau_s}.$$

The interface between the zones $x = l_1$ and $x = l_3$ is obtained using the equations

$$\frac{h\tau_s}{E_* k} \exp\Big(\frac{k\nu_*}{h}(l_3 - l)\Big) = v_*,$$

$$(1-\nu_*^2)\Big[1 + \frac{k\nu_*}{h}\Big(\frac{\coth\,(\alpha l_1)}{\alpha} + l_3 - l_1\Big)\Big] = \exp\Big(\frac{k\nu_*}{h}(l_3 - l)\Big).$$

Figure 5 shows the distribution of dimensionless contact stresses q_2 (curve 1) and p_2 (curve 2).

With plate displacement, the size of the contact zone with a constant friction force ($l_1 < x < l_3$) increases. The interface $x = l_1$ is shifted toward the center of the plate, and the interface $x = l_3$ is shifted toward the free surface.

For $k \geqslant k_p$, the interface $x = l_2$ goes beyond the plate [as in the case of the law of a constant friction force (see Sec. 2.2)]. At the first stage, when $v_* < v_{p1}$, there is only one zone, i.e., the zone of an ideal contact. With displacement of the plates, the maximum contact tangential stress increases when $x = l$ and reaches the value of τ_s for $v_* = v_{p1}$. At the same time, the second zone emerges, i.e., the zone of a constant tangential stress.

The solution obtained enables us to draw the following conclusion. Unlike the law of a constant friction force and the Amonton–Coulomb law, the two-parameter Prandtl–Il'yushin law provides a sufficiently full description of contact stresses both at high and low normal pressures.

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